On a General Formula for the Transition Temperature of Superconductors

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A method of solution of the Eliashberg equations in the theory of superconductivity is derived which uses the fact that near the transition point the energy gap is small compared to the energies over which the electron-phonon properties vary appreciably. On this basis the Eliashberg equations are converted into linear inhomogeneous integral equations. Their solution is given in operator form and provides a general formula for the transition temperature.

1. Introduction

An important objective of the theory of supercounductivity has been the calculation of the transition temperature, T_c, from the microscopic properties of the various superconducting systems. The development of the theoretical approach based on electron-phonon interactions has been reviewed recently by Bardeen[1]. In this paper we describe a reformulation of the electron-phonon interaction theory to derive analytically, a general equation for the transition temperature of a superconductor.

In 1968, McMillan[2] developed an equation for the transition temperature of a superconductor, T_c, which depends on the electron-phonon coupling constant λ, the electron-electron pseudopotential μ, and the Debye-Temperature Θ. This relation was obtained by interpolating numerical solutions of the Eliashberg equations[3, 4]. In McMillan's numerical calculations, the phonon density of states for Nb was used, which limits the application of the equation to those materials with similar phonon spectra.

Several attempts[5—10] have been made to remove this dependency of the McMillan equation on the phonon density of states of Nb, but the results are not satisfying, since they are based on either oversimplified models or an intuitive replacement of the numerical constants in the McMillan equation by functions with suitably chosen parameters.

This paper shows that by using the energy relations in a superconductor, a general formula for the transition temperature, T_c, can be derived. The non-linear equation for the gap function of the Eliashberg theory can be converted into a linear

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inhomogenous integral equation. The method is similar to that used by Enz and Quattropani[11] in solving the BCS equation for the gap function[12] for non-separable interaction kernels.

The integral equation has been solved in operator form. The T_c-formula derived in this way shows the same analytical structure as the McMillan equation, but the various constants in the latter have been replaced by functions of the electron-phonon interaction function. This paper ends with a short discussion of the results.

2. The Equation for the Gap Function

We start from the non-linear Eliashberg equations[2—4, 13]

$$\omega\left(1 - Z(\omega)\right) = \lambda \int_0^\infty d\omega' \, \text{Re}\left\{\frac{\hbar\,\omega'}{\sqrt{(\hbar\,\omega')^2 - \Delta^2(\omega')}}\right\}$$
$$\left\{K_-(\omega, -\omega') f(-\hbar\,\omega') + K_-(\omega, \omega') f(\hbar\,\omega')\right\},$$

$$\frac{1}{\hbar}\,\Delta(\omega)\,Z(\omega) = \lambda \int_0^\infty d\omega' \, \text{Re}\left\{\frac{\Delta(\omega')}{\sqrt{(\hbar\,\omega')^2 - \Delta^2(\omega')}}\right\}$$

$$\tag{2.1}$$

$$\left\{K_+(\omega, -\omega') f(-\hbar\,\omega') - K_+(\omega, \omega') f(\hbar\,\omega')\right\}$$

$$- \mu \int_0^\infty d\omega \, \text{Re}\left\{\frac{\Delta(\omega)}{\sqrt{(\hbar\,\omega)^2 - \Delta^2(\omega)}}\right\}\left(1 - 2 f(\hbar\,\omega)\right)$$

which determine the gap function $\Delta(\omega)$ and the electron renormalization function $Z(\omega)$ of a superconductor. Their kernels are

$$K_\pm(\omega, \omega') = \tfrac{1}{2} \int_0^{\omega_0} d\omega_q \, G(\omega_q)$$

$$\left\{\frac{1}{\omega' - \omega_q + \omega + i\,0} \pm \frac{1}{\omega' - \omega_q - \omega - i\,0}\right\}. \tag{2.2}$$

The density function $G(\omega_q)$ is normalized by

$$\int_0^{\omega_0} (d\omega_q/\omega_q) G(\omega_q) = 1 \qquad (2.3)$$

and is related to the electron-phonon interaction function [2, 7] using the relation

$$\tfrac{1}{2} \lambda G(\omega_q) = \alpha^2(\omega_q) F(\omega_q). \qquad (2.4)$$

The electron-phonon interaction constant, λ, is obtained by combining Eq. (2.3) and Eq. (2.4)

$$\lambda = 2 \int_0^{\omega_0} (d\omega_q/\omega_q) \alpha^2(\omega_q) F(\omega_q). $$

The direct Coulomb interaction enters into the Eq. (2.1) through the pseudopotential parameter μ and the cut-off frequency ω_c. The latter is much greater than the upper boundary frequency ω_0 of the phonon spectrum, so that for frequencies above ω_c the gap function becomes independent of ω:

$$\Delta_\infty = -\hbar \mu \int_0^{\omega_0} d\omega' \, \mathrm{Re}$$

$$\cdot \left\{ \frac{\Delta(\omega')}{\sqrt{(\hbar \omega')^2 - \Delta^2(\omega')}} \right\} (1 - 2 f(\hbar \omega')). \qquad (2.5)$$

In the Eq. (2.1), the temperature T occurs only in the Fermi distribution functions

$$f(\varepsilon) = [\exp\{\beta \varepsilon\} + 1]^{-1}$$

where $\beta = 1/k_B T$. The influence of thermal phonons has been neglected. The gap function (see appendix A) can be defined as

$$\Delta(\omega) = \Delta(0) \, \Phi_T[\Delta(0)] a(\omega) \qquad (2.6)$$

where $\Delta(0)$ is the energy gap for $\omega = 0$, and

$$\Phi_T(\Delta) = \ln \frac{\Delta}{2 \hbar \omega_1} + 2 \int_0^\infty d\varepsilon \frac{f(\sqrt{\varepsilon^2 + \Delta^2})}{\sqrt{\varepsilon^2 + \Delta^2}}. \qquad (2.7)$$

The function $a(\omega)$, which does not vanish with $\omega = 0$, satisfies the linear inhomogenous integral equation

$$Z(\omega) a(\omega) + \int_0^\infty d\omega' \ln\left(\frac{\omega'}{\omega_1}\right) \frac{\partial}{\partial \omega'}$$

$$\cdot \{\lambda K_+(\omega, \omega') - \mu^+\} \, \mathrm{Re}\,(a(\omega'))$$

$$\cdot = -(\lambda K_+(\omega, 0) - \mu^+) \qquad (2.8)$$

with a modified pseudopotential parameter

$$\mu^+ = \frac{\mu}{(1 + \mu \ln(\omega_c/\omega_1))}. \qquad (2.9)$$

The frequency ω_1 is a scaling frequency which will be determined later. For $\omega = 0$, Eq. (2.6) gives the consistency condition

$$\Phi_T(\Delta(0)) = 1/a(0) \qquad (2.10)$$

which determines the value of the energy gap.

Since the transition temperature T_c is characterized by the disappearance of the energy gap $\Delta(0)$, the renormalization function $Z(\omega)$ of the superconducting state in Eq. (2.8) can be replaced by that of the normal state. The latter results from equation (2.1) when $\Delta(0)$ is set equal to zero. Thus $Z(\omega)$ and $a(\omega)$ become analytical functions of the interaction parameters λ and μ.

By substituting $\Delta(0) = 0$ in Eq. (2.10), we get for the transition temperature (see appendix B)

$$k_B T_c = 1.135 \, \hbar \, \omega_1 \exp\{1/a(0)\}. \qquad (2.11)$$

This equation is identical to the BCS-equation for the transition temperature if $a(0)$ is replaced by $-N(0)V$. In the general case, the value of $a(\omega)$ at $\omega = 0$ assumes the function of the BCS coupling constant [12]. The problem of the transition temperature is, therefore, reduced to the calculation of $a(0)$.

3. Solution of the Linear Integral Equation

We consider first, the pure phonon case ($\mu = 0$). The solution of Eq. (2.8) can be written in the form:

$$a_{\mathrm{Ph}}(\omega) = a_{\mathrm{Ph}}(0) b_{\mathrm{Ph}}(\omega) \qquad (3.1)$$

with $b_{\mathrm{Ph}}(0) = 1$. The function $b_{\mathrm{Ph}}(\omega)$ satisfies the equation

$$Z(\omega) b_{\mathrm{Ph}}(\omega) - \lambda \int_0^\infty \frac{d\omega'}{\omega'} M(\omega, \omega') \mathrm{Re}\,(b_{\mathrm{Ph}}(\omega'))$$

$$= (1 + \lambda) K_+(\omega, 0) \qquad (3.2)$$

where

$$M(\omega, \omega') = K_+(\omega, \omega') - K_+(\omega, 0) K(0, \omega)$$

and the value $a(0)$ is given by means of $b_{\mathrm{Ph}}(\omega)$ by the relation

$$\frac{1}{a_{\mathrm{Ph}}(0)} = \frac{1 + \lambda}{\lambda} (1 + A)$$

$$A = \frac{\lambda}{1 + \lambda} \int_0^\infty d\omega \ln\left(\frac{\omega}{\omega_1}\right) \frac{\partial}{\partial \omega}$$

$$\cdot \{K_+(0, \omega) \mathrm{Re}\,(b_{\mathrm{Ph}}(\omega))\}. \qquad (3.3)$$

In deriving this equation the conditions

$$Z(0) = 1 + \lambda, \qquad K_+(0, 0) = 1 \qquad (3.4)$$

have been used.

If the integral in Eq. (3.2) is replaced by an operator $\overline{M}$

$$M g(\omega) = \frac{1+\lambda}{Z(\omega)} \int\limits_0^\infty \frac{d\omega'}{\omega'} M(\omega, \omega') \, \text{Re}\, \big(g(\omega')\big) \quad (3.5)$$

$[g(\omega) -$ any reasonable function], then the solution of (3.2) can be written in operator form as

$$b_{\text{Ph}}(\omega) = \left(1 - \frac{\lambda}{1+\lambda} \overline{M}\right)^{-1} \left(\frac{1+\lambda}{Z(\omega)} K_+(\omega, 0)\right). \quad (3.6)$$

Since the renormalization function of the normal state can be expressed as

$$\frac{Z(\omega)}{1+\lambda} = 1 + \frac{\lambda}{1+\lambda} \zeta(\omega), \quad (3.7)$$

where $\zeta(\omega)$ no longer depends on λ, and goes to zero when $\omega = 0$, the electron-phonon interaction parameter, λ, enters into Eq. (3.6) only through the ratio $\lambda/(1+\lambda)$. Furthermore, the solution $b_{\text{Ph}}(\omega)$ does not depend on ω_1, as can be seen from Eq. (3.2) or Eq. (3.6).

The general solution $(\mu = 0)$ is obtained by splitting up the function for $a(\omega)$

$$a(\omega) = a_{\text{Ph}}(\omega) + \mu^+ a_{\text{Cb}}(0) b_{\text{Cb}}(\omega) \quad (3.8)$$

with $b_{\text{Cb}}(0) = 1$. Here $b_{\text{Cb}}(\omega)$ satisfies the equation

$$Z(\omega) b_{\text{Cb}}(\omega) + \lambda \int\limits_0^\infty d\omega' \ln\left(\frac{\omega'}{\omega_1}\right) \frac{\partial}{\partial\omega'} \quad (3.9)$$
$$\cdot \big\{ \big(K_+(\omega, \omega') - K_+(0, \omega')\big) \, \text{Re}\, \big(b_{\text{Cb}}(\omega')\big) \big\} = 1 + \lambda$$

while the value of $a_{\text{Cb}}(\omega)$ at $\omega = 0$ is given by

$$\frac{1}{a_{\text{Cb}}(0)} = \frac{1 + \lambda + \lambda B_0 - \mu^+ B_{\text{Cb}}}{1 + a_{\text{Ph}}(0) B_{\text{Ph}}} \quad (3.10)$$

and the constants

$$B_{\text{Ph, Cb}} = \int\limits_0^{\infty,\, \omega_c} d\omega \ln\left(\frac{\omega}{\omega_1}\right) \frac{\partial}{\partial\omega} \{\text{Re}\,(b_{\text{Ph, Cb}}(\omega))\}$$
$$B_0 = \int\limits_0^\infty d\omega \ln\left(\frac{\omega}{\omega_1}\right) \frac{\partial}{\partial\omega} \{K_+(0, \omega) \, \text{Re}\,(b_{\text{Cb}}(\omega))\}$$
$$(3.11)$$

are defined by the functions $b_{\text{Ph}}(\omega)$ and $b_{\text{Cb}}(\omega)$. Since Eq. (3.9) does not contain the parameter μ^+, the solution $b_{\text{Cb}}(\omega)$ and the constants (3.11) are independent of the Coulomb interaction.

With the operator $\overline{M}$ the solution of (3.9) can finally be rewritten as

$$b_{\text{Cb}}(\omega) = \left(1 + \frac{\lambda}{1+\lambda} B_0\right)\left(1 - \frac{\lambda}{1+\lambda} \overline{M}\right)^{-1}$$
$$\cdot \left\{\frac{1+\lambda}{Z(\omega)} (1 - K_+(\omega, 0))\right\}. \quad (3.12)$$

In the next section it will be shown that B_0 depends on the electron-phonon interaction parameter only through the ratio $\lambda/(1+\lambda)$, and by choosing a suitable value of ω_1, no longer contains this scaling frequency. Thus the Coulomb function (3.12) is also a function only of the ratio $\lambda/(1+\lambda)$ and the scaling frequency ω_1.

From the Eq. (3.8), (3.3) and (3.10) we finally get the effective coupling constant

$$a(0) = -\frac{\lambda - (1+C)\mu^*}{(1+\lambda)(1+A)} \quad (3.13)$$

with

$$1 + C = \frac{\left(1 + A - \frac{\lambda}{1+\lambda} B_{\text{Ph}}\right)}{\left(1 + \frac{\lambda}{1+\lambda} B_0\right)} \quad (3.14)$$

and the remodified Coulomb pseudopotential parameter

$$\mu^* = \frac{\mu}{1 + \frac{\mu}{1+\lambda} \ln \frac{\omega_c}{\omega_2}},$$

$$\ln \frac{\omega_c}{\omega_2} = (1+\lambda) \ln \frac{\omega_c}{\omega_1} - \frac{B_{\text{Cb}}}{1 + \frac{\lambda}{1+\lambda} B_0}. \quad (3.15)$$

4. The Transition Temperature T_c

To obtain the analytical structure of the formula for the transition temperature, the following must be shown:

1) that the scaling frequency ω_1 can be chosen in such a way that all values entering into the effective coupling constant (3.13) do not depend on it, and

2) that B_0 in Eq. (3.11) or Eq. (3.12) is a function only of the ratio $\lambda/(1+\lambda)$.

The latter can be seen directly, if the formal solution (3.12) is inserted in Eq. (3.11). The resulting consistency relation can be solved for B_0 using the definition (3.3) of the parameter A and the solution (3.6) for the pure phonon case.

This gives

$$B_0 = \frac{D}{1 + A - \dfrac{\lambda}{1 + \lambda} D} \tag{4.1}$$

with

$$D = \int_0^\infty d\omega \, \ln\left(\frac{\omega}{\omega_1}\right) \frac{\partial}{\partial \omega} \left\{ K_+(0, \omega) \operatorname{Re}\left[\left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 + \lambda}{Z(\omega)}\right]\right\}. \tag{4.2}$$

From this representation, we see that both, D and B_0 contain only the ratio $\lambda/(1 + \lambda)$.

If the above expression for B_0 is substituted into (3.14), then the parameter $1 + C$ can be rewritten as

$$1 + C = \frac{\left(1 - \dfrac{\lambda}{1 + \lambda} B_1\right)\left(1 - \dfrac{\lambda}{1 + \lambda} B_2\right)}{(1 + A)}$$

where the new constants are defined by the integrals

$$B_1 = \int_0^\infty \frac{d\omega}{\omega} K_+(0, \omega) \operatorname{Re}\left[\left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 + \lambda}{Z(\omega)} \left(K_+(\omega, 0) - 1\right)\right],$$

$$B_2 = \int_0^\infty \frac{d\omega}{\omega} \left(K_+(0, \omega) - 1\right) \operatorname{Re}\left[\left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 + \lambda}{Z(\omega)} K_+(\omega, 0)\right]. \tag{4.4}$$

Finally, with solution (3.12) from the definition (3.11) of B_{Cb} we get

$$\frac{B_{\mathrm{Cb}}}{\left(1 + \dfrac{\lambda}{1 + \lambda} B_0\right)} = \int_0^{\omega_c} d\omega \, \ln\left(\frac{\omega}{\omega_1}\right) \frac{\partial}{\partial \omega} \left\{\operatorname{Re}\left[\left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 + \lambda}{Z(\omega)} \left(1 - K_+(\omega, 0)\right)\right]\right\}.$$

Integrating by parts, the term $(1 + \lambda) \ln(\omega_c/\omega_1)$ can be extracted so that the frequency ω_2 is actually defined by

$$\ln\frac{\omega_c}{\omega_2} = \int_0^{\omega_c} \frac{d\omega}{\omega} \operatorname{Re}\left[\left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 + \lambda}{Z(\omega)} \left(1 - K_+(\omega, 0)\right)\right]. \tag{4.5}$$

The scaling frequency ω_1, therefore, enters the effective coupling constant only through the parameter A, which with the solution (3.6) can be written as

$$A = \frac{\lambda}{1 + \lambda} \int_0^\infty d\omega \, \ln\left(\frac{\omega}{\omega_1}\right) \frac{\partial}{\partial \omega} \left\{ K_+(0, \omega) \operatorname{Re}\left[\left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 + \lambda}{Z(\omega)} K_+(\omega, 0)\right]\right\}. \tag{4.6}$$

It is also independent of ω_1, if we define the scaling frequency by

$$\ln\omega_1 = -\int_0^\infty d\omega \, \ln\omega \, \frac{\partial}{\partial \omega} \left\{ K_+(0, \omega) \operatorname{Re}\left(K_+(\omega, 0)\right)\right\} \tag{4.7}$$

since (4.6) is then converted into

$$A = \frac{\lambda}{1 + \lambda} \int_0^\infty \frac{d\omega}{\omega} K_+(0, \omega) \operatorname{Re}\left[K_+(\omega, 0) - \left(1 - \frac{\lambda}{1 + \lambda} \overline{M}\right)^{-1} \frac{1 - \lambda}{Z(\omega)} K_+(\omega, 0)\right]. \tag{4.8}$$

With this choice of ω_1 Eq. (4.2) gives an integral for D, independent of ω_1

$$D = \int\limits_0^\infty \frac{d\omega}{\omega} K_+(0,\omega) \operatorname{Re}\left[K_+(\omega,0) - \left(1 - \frac{\lambda}{1+\lambda}\,\overline{M}\right)^{-1} \frac{1+\lambda}{Z(\omega)} \right] \qquad (4.9)$$

and one obtains an effective coupling constant independent of ω_1, so that together with Eq. (2.11) and (3.13) the following general formula for the transition temperature results:

$$k_B T_c = 1.135\,\hbar\,\omega_1 \exp\left\{ -\frac{(1+\lambda)(1+A)}{\lambda - (1+C)\mu^*} \right\}. \qquad (4.10)$$

Besides λ and μ^*, the various constants defined by means of Eqs. (4.7), (4.8), (4.3) and (4.4) are functions of the kernel $K_+(\omega,\omega')$ as given in (2.2), and of the operator $\overline{M}$ defined by (3.2). They are therefore analytical functions of $\lambda/(1+\lambda)$.

5. Discussion

The non-linear Eliashberg-equation can be transformed into the linear inhomogenous integral Eq. (2.8), because the energy gap in a superconducting metal is small in comparison with the characteristic phonon energies. Since all superconductors fulfill this condition at least in the vicinity of their transition points, the new equation should be valid in this temperature region. An advantage is that its solution can be given in closed operator form, in which the electron-phonon coupling parameter λ appears in the various functions only in the ratio $\lambda/(1+\lambda)$. The denominator represents the influence of the electron-phonon renormalization, by which the bare coupling parameter is reduced to an effective one. This technique may be useful for further approximations, since the ratio $\lambda/(1+\lambda)$ varies only from zero to one.

A general formula (4.1) for the transition temperature of a superconductor is derived without any approximations from the operator solutions. It relates the transition temperature to a scaling frequency and several other functions, depending on the normalized electron-phonon interaction functions via the kernels (2.2). This T_c-expression generalizes the interpolation formula of McMillan[2] and, since its derivation is purely analytical, also gives a strong justification for his interpolation. In addition, the new equation shows the connection between the diverse constants of the T_c-formula and the interaction function. It can, therefore, be used to investigate the effect of changes in the normalized electron-phonon interaction function on the calculated transition temperature of a superconductor.

In Table 1 values of the transition temperature calculated from equation (4.1) are compared with

Table 1. Comparison of measured transition temperatures of thin metal films with calculated values [15].

Metal *	T_c measured	T_c McMillan	T_c Garland	T_c Eq. (4.10)
Sn, 300 K	3,6	4,8	2,3	4,4
Sn, 2 K	4,5	10,2	4,2	5,4
Sn+10% Cu, 2 K	6,8	24,0	6,2	6,9
Pb, 300 K	7,2	10,3	5,8	7,4
Pb, 2 K	7,2	12,7	5,7	6,5
Pb+10% Cu, 2 K	6,5	14,2	5,2	4,8

* This column gives the composition and the condensation or annealing temperature of the various metal films investigated by Knorr and Barth [16].

experimentally determined values of Knorr and Barth [15] as well as those calculated from McMillan's [2] formula and one due to Garland [7]. The calculation had been done without adjusting any parameter. The difference between the measured values and our calculated ones are believed to result from approximations necessary to carry out the calculations and the inaccuracy of the experimental electron-phonon interaction functions. Details of the calculations will be given elsewhere [15].

Recently, Kirshnitz, Maksimow and Khomskii [14] derived a similar equation for the transition temperature of a superconductor. Their approach is based on the description of superconductivity in terms of the dielectric response function of the normal system. Since these authors neglect the electron-phonon renomalization, their expression does not depend on the effective but on the bare coupling parameter λ. They give, however, a definition of the scaling frequency ω_1, which is the same as the one derived here, since ω_1 is independent of λ.

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Appendices

A) Derivation of the Linear Inhomogenous Integral Equation

The gap function can be split into two terms

$$\Delta(\omega) = \Delta_0 \bar{a}(\omega)$$

where Δ_0 is the value of the gap function for $\omega = 0$. Putting this into the second Eliashberg equation for the pure phonon case ($\mu = 0$) gives

$$\bar{a}(\omega) Z(\omega) = \lambda \hbar \int_0^\infty d\omega' \, \mathrm{Re} \left\{ \frac{\bar{a}(\omega')}{\sqrt{(\hbar\,\omega')^2 - \Delta_0^2 \bar{a}(\omega')^2}} \right\}$$
$$* \left\{ f(-\hbar\,\omega') K_+(\omega, -\omega') - f(\hbar\,\omega') K_+(\omega, \omega') \right\}.$$
$$\text{(A1)}$$

We divide the kernel into the symetric and antisymetric parts with respect to ω'

$$K_+(\omega, \omega') = K_s(\omega, \omega') + \hbar\,\omega' K_a(\omega, \omega')$$
$$K_s(\omega, \omega') = \tfrac{1}{2} \left(K_+(\omega, \omega') + K_+(\omega_1 - \omega') \right)$$
$$\hbar\,\omega' K_a(\omega, \omega') = \tfrac{1}{2} \left(K_+(\omega, \omega') - K_+(\omega, -\omega') \right),$$
$$\text{(A2)}$$

where $K_s(\omega, \omega')$ and $K_a(\omega, \omega')$ are symetric and slowly varying functions of ω' near $\omega' = 0$. If δ is chosen, so that the symetric and antisymetric parts can be considered as constant for $0 < \omega' < \delta$, and if δ is many times greater than Δ_0, the integration in (A1) can be written

$$\bar{a}(\omega) Z(\omega) - \lambda \int_\delta^\infty \frac{d\omega'}{\omega'} K_+(\omega, -\omega') \, \mathrm{Re} \{ \bar{a}(\omega') \}$$
$$= \lambda \hbar \int_0^\delta d\omega' \, \mathrm{Re} \left\{ \frac{\bar{a}(\omega')}{\sqrt{(\hbar\,\omega')^2 - \Delta_0^2 \bar{a}(\omega')^2}} \right\}$$
$$\left\{ K_s(\omega, 0) \left(1 - 1 f(\hbar\,\omega') \right) - \hbar\,\omega' K_a(\omega, 0) \right\}. \quad \text{(A3)}$$

In the first integral, the Fermi distribution function has been neglected taking into account the low temperatures, whereas the square root has been replaced by $\hbar\,\omega'$ because Δ_0 is very much smaller than δ. The main contribution to the integral of the right

hand side comes from the singularity of the integrand. Since we are interested in the limit $\Delta_0 \to 0$. we can put $a(0) = 1$ everywhere insteat of $\bar{a}(\omega')$, if we notice that $\bar{a}(\omega')$ is a symetric function in ω'.

The integration on the right hand side can be done and gives, remembering that $\Delta_0 \ll \delta$:

$$\bar{a}(\omega) Z(\omega) - \lambda \int_0^\infty \frac{d\omega'}{\omega'} K_+(\omega, -\omega') \, \mathrm{Re} \{ \bar{a}(\omega') \}$$
$$= -\lambda K_s(\omega, 0) \Phi_T(\Delta_0) + \lambda \{ K_s(\omega, 0) \ln \delta$$
$$- \hbar \, \delta K_a(\omega, 0) \}$$
$$\Phi_T(\Delta_0) = \ln \frac{\Delta_0}{2\hbar} + 2 \int_0^\infty d\varepsilon \, \frac{f(\sqrt{\varepsilon^2 + \Delta_0^2})}{\sqrt{\varepsilon^2 + \Delta_0^2}} . \quad \text{(A4)}$$

The remaining integral can be transformed as follows:

$$\int_\delta^\infty \frac{d\omega'}{\omega'} \left(K_s(\omega, +\omega') - \hbar\,\omega' K_a(\omega, \omega') \right) \mathrm{Re}\, \bar{a}(\omega')$$

$$= \left[\int_0^\infty \frac{d\omega'}{\omega'} - \int_0^\delta \frac{d\omega'}{\omega'} \right] \{ \left(K_s(\omega, \omega') - \hbar\,\omega' K_a(\omega, \omega') \right)$$
$$\cdot \mathrm{Re}\, \bar{a}(\omega') \}$$
$$= \{ \ln \omega' \left(K_s(\omega, \omega') + \hbar\,\omega' K_a(\omega, \omega') \right) \mathrm{Re}\, \bar{a}(\omega') \} \big|_0^\infty$$
$$- \int_0^\infty d\omega' \ln \omega' \frac{\partial}{\partial \omega'} \left(K_+(\omega, -\omega') \mathrm{Re}\, \bar{a}(\omega') \right)$$
$$- K_s(\omega, 0) \ln \omega' \big/_0^\delta + \hbar \, \delta K_a(\omega, 0)$$
$$= - \int_0^\infty d\omega' \ln \omega' \frac{\partial}{\partial \omega} \left(K_+(\omega, -\omega') \mathrm{Re}\, \bar{a}(\omega') \right)$$
$$- \left(K_s(\omega, 0) \ln \delta - \hbar \, \delta K_a(\omega, 0) \right)$$

and the above equation transforms into

$$\bar{a}(\omega) Z(\omega) + \lambda \int_0^\infty d\omega' \ln \frac{\omega'}{\omega_1} \frac{\partial}{\partial \omega}$$
$$\cdot \left(K_+(\omega, -\omega') \mathrm{Re} \{ \bar{a}(\omega') \} \right)$$
$$= -\lambda K_s(\omega, 0) \Phi_T(\Delta_0)$$
$$\Phi_T(\Delta_0) = \ln \frac{\Delta_0}{2\hbar\,\omega_1} + 2 \int_0^\infty d\varepsilon \, \frac{f(\sqrt{\varepsilon^2 + \Delta_0^2})}{\sqrt{\varepsilon^2 + \Delta_0^2}} \quad \text{(A5)}$$

which is independent of δ. The frequency ω_1, which is a scaling frequency, has been introduced for convenience and will be specified later. The case including the Coulomb repulsions results if we take into account the value Δ_∞ of the gap functions at

high frequency [see Equation (2.5)]. That means adding

$$\bar{a}_\infty = \frac{\Delta_\infty}{\Delta_0} = - \mu \, \hbar \int_0^{\omega_c} d\omega' \, \text{Re}$$

$$\cdot \left\{ \frac{a(\omega')}{\sqrt{(\hbar \, \omega')^2 - \Delta_0{}^2 a^2(\omega)}} \right\} (1 - 2 f(\hbar \, \omega'))$$

to the right hand side of (A5).

With the same method as used for the pure phonon case this can be transformed into

$$\bar{a}_\infty = \mu^+ \, \Phi_T(\Delta_0) - \mu^+ \int_0^{\omega_c} d\omega' \ln \frac{\omega'}{\omega_1} \frac{\partial}{\partial \omega'} \, \text{Re} \, \bar{a}(\omega') \tag{A6}$$

where μ^+ denotes the modified Coulomb pseudopotential

$$\mu^+ = \mu / [1 + \mu \ln(\omega_c/\omega_1)] \tag{A7}$$

We, therefore, arrive at the equation

$$\bar{a}(\omega) Z(\omega) + \int_0^\infty d\omega' \frac{\partial}{\partial \omega'} \left(\lambda K_+(\omega, -\omega') - \mu^+ \right)$$

$$\cdot \, \text{Re} \{ \bar{a}(\omega') \}$$

$$= -\left(\lambda K_+(\omega, 0) - \mu^+ \right) \Phi_T(\Delta_0) . \tag{A8}$$

Here it is tacitly understood that the integration in the part containing μ^+ is only extended to the cut-off frequency ω_c. This equation is linear in the unknown function $\bar{a}(\omega)$ and the factor $\Phi_T(\Delta_0)$ can be split off. Altogether, the gap function can be written in the form of Eq. (2.6) as

$$\Delta(\omega) = \Delta(0) \, \Phi_T(\Delta(0)) \, a(\omega)$$

where $a(\omega)$ is a solution of Eq. (2.8) which results from (A8) if $\Phi_T(\Delta_0)$ is replaced by one.

B) The Function $\Phi_T(\Delta_0)$

In Sect. 2 the function $\Phi_T(\Delta_0)$ is needed which essentially determines the temperature for the superconducting transition. It can be reduced to a function defined by Mühlschlegel [17], if the term

$$\Phi_T(0) = \ln\left(\frac{\pi}{2 \gamma}\right) \frac{K_B T}{\hbar \, \omega_1} \tag{B1}$$

is split off:

$$\Phi_T(\Delta_0) = \Phi_T(0) + \eta(\Delta_B / K_B T) . \tag{B2}$$

The function $\eta(x)$ is defined by

$$\eta(x) = \ln(\gamma x) + \frac{2}{\pi} \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} dt \ln(1 - \exp\{\pi \sqrt{t^2 + x^2}\})$$

$$= 2 \sum_{K=1}^\infty \left(\frac{1}{2 K - 1} - \frac{1}{\sqrt{(2 K - 1)^2 + x^2}} \right) \tag{B3}$$

where $\gamma = 0.5772 \ldots$ is Euler's constant. (B3) shows that for positive x, $\eta(x)$ is a monotonically increasing function of the argument and for small x-values can be approximated by

$$\eta(x) = \tfrac{7}{8} \zeta(3) x_2 , \tag{B4}$$

with Riemann's ζ-function. Equation (2.10), therefore, possesses just one solution as long as $\Phi_T(0)$ is not greater than $a(0)^{-1}$. So the transition temperature is determined by the relation (2.11) given in the text.

[1] J. Bardeen, Physics today **26**, 41 [July 1973].
[2] W. L. McMillan, Phys. Rev. **167**, 331 [1968].
[3] G. M. Eliashberg, Sov. Phys. JETP **12** 1000 [1961].
[4] D. J. Scalapino in: Superconductivity ed. by R. D. Parks, New York 1969, Vol. I, p. 500.
[5] J. W. Garland, K. H. Bennemann, and F. M. Mueller, Phys. Rev. Letters **21**, 1315 [1968].
[6] R. Benda, Phys. Stat. Sol. **38**, 451 [1969].
[7] J. W. Garland, and P. B. Allen, Physica **55**, 669 [1971].
[8] P. Hertel, Z. Phys. **248**, 272 [1971].
[9] V. T. Geilikman, and N. F. Masharow, J. Low Temp. Physics **6**, 131 [1972].
[10] V. Kressin, Sov. Phys. Solid State **13**, 2468 [1972].
[11] C. P. Enz and A. Quattropani, Helv. phys. Acta **41**, 54 [1969].
[12] J. Bardeen, L. N. Cooper, and J. R. Schrieffer. Phys. Rev. **108**, 1175 [1957].
[13] J. G. Swihardt, D. J. Scalapino, and Y. Wada, Phys. Rev. Lett. **14**, 106 [1965].
[14] D. A. Kirzhnitz, E. G. Maksimow, and D. I. Khonskii, J. Low Temp. Phys. **10**, 79 [1973].
[15] The values in Table 1 are the values which the author has given in his doctoral thesis, Braunschweig 1972, and the internal report: PTB-Bericht-W-1-W2, 1973.
[16] K. Knorr, and N. Barth, Solid State Com. **8**, 1085 [1970].
[17] B. Mühlschlegel, Z. Phys. **155**, 313 [1959].